A note on the well-posedness of non-autonomous linear evolution equations

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We investigate a commonly used theorem by Yosida (Theorem XIV.4.1 of Yosida's book on functional analyis or Theorem X.70 of Reed and Simon's book on mathematical physics) on the well-posedness of the initial value problems corresponding to a family A of linear operators $A(t)$ with common dense domain D in some Banach space X. We prove that the rather long regularity conditions of this theorem can be replaced – without changing the content of the theorem – by the single (and simple) condition that $t \mapsto A(t)x$ is continuously differentiable for every $x \in D$. We also extend this simplification to the case of sequentially complete locally convex spaces X. With these observations we finally clarify the relation between Yosida's theorem and other classical theorems on well-posedness.

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1 Introduction

In this note we shall be concerned with the well-posedness of non-autonomous linear evolution equations, that is, initial value problems of the following kind:

$$x' = A(t)x, \ x(s) = y,$$

where $A(t) : D \subset X \to X$, for every $t \in [0, 1]$, is a linear operator in a Banach space X with a dense domain D that is independent of t and where the initial value y lies in D and the initial time s is in $[0, 1)$. A simple sufficient condition for these initial value problems to be well-posed is that $A(t)$ generates a contraction semigroup in X for every $t \in [0, 1]$ and that $t \mapsto A(t)x$ is continuously differentiable for every $x \in D$ (which is due to Kato (Theorem 4 of [9]) and is reproduced, for instance, in Theorem 5.4.8 of [17]).

In a widely used theorem by Yosida (Theorem XIV.4.1 of [27]) – which is also to be found in [18] (Theorem X.70) or in [6] (Theorem 9.5.3) – sufficient conditions are given that seem to be more general than the simple conditions just mentioned. We will show in this note that, actually, they are not. We will prove (in Theorem 6) that the sufficient conditions of Yosida's theorem from [27] are fulfilled if and only if the abovementioned simple sufficient conditions are fulfilled and $A(t)$ is boundedly invertible for every $t \in [0,1]$. An immediate consequence is that Yosida's theorem is equivalent to Kato's theorem above.

We also include in our considerations Yosida's article [26], where well-posedness is established for the case of linear operators $A(t) : D \subset X \to X$ with common dense domain D in a locally convex space X, and we extend the equivalence just mentioned to this more general case (in Theorem 9).

We obtained these equivalences in the context of adiabatic theory (in the work [21]): many adiabatic theorems in the literature – for example those of [5], [3], [24], [25], [1], [2] or [4] – use the long conditions given in [27] or [18] or more convenient strengthenings thereof in order to guarantee well-posedness. It follows from the equivalences established in this note that the regularity conditions of these adiabatic theorems can be remarkably simplified (from the theoretical and practical point of view) without making them less general, and sometimes – for example in the case of the very recent paper [4] – making them even more general.

A word on the notation used in this note: if not explicitly stated otherwise, X will denote a Banach space over $\mathbb{C}$. As regards terminological conventions for locally convex spaces we refer to Schaefer's book [20] (in particular, a locally convex space will always be assumed to be Hausdorff).

2 Some preliminaries

We begin our exposition with a lemma that will be crucial to the simplification we aim at. It is a variation of a well-known lemma on the relation between differentiability and left or right differentiability (Lemma III.1.36 of [12]).

Lemma 1. *(i) Suppose $f : [0,1] \to X$ is a continuous map such that the limit $g(t) := \lim_{k\to\infty} -k\left(f(t-\frac{1}{k}) - f(t)\right)$ exists uniformly in $t \in (0,1]$, that is, the limit exists for every $t \in (0,1]$ and $\sup_{t\in[\frac{1}{k},1]} \left\|-k\left(f(t-\frac{1}{k}) - f(t)\right) - g(t)\right\| \longrightarrow 0$ $(k \to \infty)$. Then*

$$\|f(t) - f(0)\| \le \sup_{\tau\in(0,1)} \|g(\tau)\| \, (t-0) \quad \textit{for all } t \in [0,1].$$

(ii) Suppose, in addition, that g is continuous and continuously extendable to the left endpoint 0 of the unit interval. Then f is continuously differentiable on $[0,1]$ and $f' = g$.

Proof. (i) Set $M := \sup_{\tau\in(0,1)} \|g(\tau)\|$. As a first step we show that for every $\varepsilon \in (0,1]$

there is an $a_\varepsilon \in (0,1)$ such that

$$\|f(a_\varepsilon) - f(0)\| < \big(M + \varepsilon\big) a_\varepsilon$$

and that these a_ε can be chosen to decrease to 0 as ε decreases to 0. In virtue of the uniform convergence assumption, we can choose, for every $\varepsilon \in (0,1]$, a natural number $k_{1\,\varepsilon} \in \mathbb{N}$ minimal with the property that $\sup_{t\in[\frac{1}{k},1]} \left\| -k\left(f(t-\frac{1}{k}) - f(t)\right) - g(t)\right\| < \varepsilon$ for all $k \geq k_{1\,\varepsilon}$. Setting $a_\varepsilon := \frac{1}{k_{1\,\varepsilon}+l_\varepsilon} \in (0,1)$ with $l_\varepsilon := \lfloor \frac{1}{\varepsilon} \rfloor$, we see that

$$\begin{aligned}
\|f(a_\varepsilon) - f(0)\| &= \left\| f(a_\varepsilon) - f\big(a_\varepsilon - \frac{1}{k_{1\,\varepsilon} + l_\varepsilon}\big) \right\| \\
&\leq \left\| -(k_{1\,\varepsilon} + l_\varepsilon)\left(f\big(a_\varepsilon - \frac{1}{k_{1\,\varepsilon} + l_\varepsilon}\big) - f(a_\varepsilon)\right) - g(a_\varepsilon) \right\| \frac{1}{k_{1\,\varepsilon} + l_\varepsilon} + \|g(a_\varepsilon)\| \frac{1}{k_{1\,\varepsilon} + l_\varepsilon} \\
&< (M+\varepsilon) a_\varepsilon
\end{aligned}$$

and that a_ε decreases to 0 as ε decreases to 0 (due to the minimality of $k_{1\,\varepsilon}$). As a second step we show that for every $\varepsilon \in (0,1]$ the bad set

$$J_\varepsilon := \{t \in [a_\varepsilon, 1] : \|f(t) - f(0)\| > \big(M+\varepsilon\big)(t-0)\}$$

is empty. Assuming the contrary, we get that the infimum $t_\varepsilon := \inf J_\varepsilon$ lies in $[a_\varepsilon, 1]$ and, using the continuity of f, we see that t_ε must be different from a_ε and from 1. We can therefore choose a natural number $k_{2\,\varepsilon} \in \mathbb{N}$ such that $a_\varepsilon \leq t_\varepsilon - \frac{1}{k_{2\,\varepsilon}}$ and $t_\varepsilon + \frac{1}{k_{2\,\varepsilon}} \leq 1$. So, if we set $k_\varepsilon := \max\{k_{1\,\varepsilon}, k_{2\,\varepsilon}\}$, we get

$$\begin{aligned}
\|f(t) - f(0)\| &\leq \left\| f(t) - f\big(t - \frac{1}{k_\varepsilon}\big)\right\| + \left\| f\big(t - \frac{1}{k_\varepsilon}\big) - f(0)\right\| \\
&\leq \left\| -k_\varepsilon\left(f\big(t-\frac{1}{k_\varepsilon}\big) - f(t)\right) - g(t)\right\| \frac{1}{k_\varepsilon} + \|g(t)\| \frac{1}{k_\varepsilon} + \left\| f\big(t-\frac{1}{k_\varepsilon}\big) - f(0)\right\| \\
&\leq (M+\varepsilon)\frac{1}{k_\varepsilon} + (M+\varepsilon)(t - \frac{1}{k_\varepsilon} - 0) = (M+\varepsilon)(t-0)
\end{aligned}$$

for all $t \in [t_\varepsilon, t_\varepsilon + \frac{1}{k_\varepsilon})$. And from this it follows that $t_\varepsilon = \inf J_\varepsilon \geq t_\varepsilon + \frac{1}{k_\varepsilon}$, which is impossible. We have thus shown that J_ε must be empty, and since a_ε decreases to 0 as ε decreases to 0, the desired conclusion follows by letting ε tend to 0.

(ii) Set $h(t) := f(t) - \int_{t_0}^t g(\tau)\,d\tau$ for $t \in [0,1]$ where t_0 is an arbitrary element of $[0,1]$. It is then immediately clear that h is a continuous map $[0,1] \to X$ and that

$$-k\left(h(t-\frac{1}{k}) - h(t)\right) \longrightarrow 0 \quad (k \to \infty)$$

uniformly in $t \in (0,1]$. And therefore, h is constant by virtue of part (i). So we have

$$f(t) = f(t_0) + \int_{t_0}^t g(\tau)\,d\tau$$

for all $t \in I$, from which the assertion is obvious. ■

Completely analogous arguments show that the above lemma carries over to the case where X is a sequentially complete locally convex space (over $\mathbb{C}$): one has only to replace norms by continuous seminorms (and, for the constancy of h in the proof of part (ii), to remember that we assume locally convex spaces to be Hausdorff). In the mean value estimate of part (i), it would suffice, of course, to take the supremum only over the t-dependent interval $(0,t)$, but this will not be needed in the sequel.

It is now time to properly define the concept of well-posedness and the closely related concept of evolution systems. In the following we shall write J for a non-trivial compact interval $[a,b]$ and Δ_J for the triangular set $\{(s,t) \in J^2 : s \le t\}$ associated with the interval J.

Suppose that $A(t) : D(A(t)) \subset X \to X$ is a linear operator for all $t \in J$ and that Y is a dense subspace of X such that $Y \subset D(A(t))$ for all $t \in J$. A mapping u is then said to be a Y-*solution* to (or to Y-*solve*) the initial value problem

$$x' = A(t)x, \; x(s) = y$$

on the interval $[s,b]$ (where $y \in Y$ and $s \in [a,b)$) if and only if $u \in C^1([s,b],X)$, u maps $[s,b]$ into Y, and $u'(t) = A(t)u(t)$ for all $t \in [s,b]$ and $u(s) = y$. Additionally, the initial value problems corresponding to the family A are called *well-posed on the space* Y if and only if

(i) the initial value problem

$$x' = A(t)x, \; x(s) = y$$

is uniquely Y-solvable on $[s,b]$ for all $y \in Y$ and all $s \in [a,b)$ (with unique Y-solution $x(\,.\,,s,y)$)

(ii) $\Delta_J \ni (s,t) \mapsto x(t,s,y)$ is continuous for all $y \in Y$ and $x(t,s,y_n)$ converges to 0 uniformly in $(s,t) \in \Delta_J$ for all sequences (y_n) in Y with $y_n \to 0$ as $n \to \infty$, where $x(b,b,y) = y$ by definition.

Suppose that $A(t) : D(A(t)) \subset X \to X$ is a linear operator for every $t \in J$ and that $U(t,s)$ is a bounded linear operator in X for every $(s,t) \in \Delta_J$. Suppose further that Y is a dense subspace of X such that $Y \subset D(A(t))$ for all $t \in J$. U is then called *an evolution system for* A *on the space* Y if and only if the following holds true:

(i) $[s,b] \ni t \mapsto U(t,s)x$ Y-solves the initial value problem

$$x' = A(t)x, \; x(s) = y$$

for all $y \in Y$ and all $s \in [a,b)$, and $U(t,s)U(s,r) = U(t,r)$ for all $(r,s),(s,t) \in \Delta_J$

(ii) $\Delta_J \ni (s,t) \mapsto U(t,s)x$ is continuous for all $x \in X$.

We explicitly point out that a Y-solution is required to be continuously differentiable and not only differentiable (which would be the minimal requirement). It turns out that the major classical theorems on well-posedness of linear evolution equations – a few of them will be touched upon in Section 4 – all guarantee continuous differentiability whence it is reasonable (and common) to incorporate it into the definition of solutions. A definition of well-posedness and evolution systems that covers also the case of variable subspaces Y_t can be found in Section VI.9 of [7] and can readily be seen to contain the above definition as a special case.

The following well-known proposition (see Proposition VI.9.3 of [7]) reveals the close relation between evolution systems and well-posedness. In particular, it immediately follows from this proposition that, if there is any evolution system on Y for a given family A of linear operators (with $Y \subset D(A(t))$), then it is already unique.

Proposition 2. *Suppose that $A(t) : D(A(t)) \subset X \to X$ is a linear operator for every $t \in J = [a, b]$ and that Y is a dense subspace of X such that $Y \subset D(A(t))$ for all $t \in J$. Then the initial value problems corresponding to A are well-posed on Y if and only if there exists an evolution system for A on Y.*

In the case of locally convex spaces X the definition of Y-solutions can be taken over literally, but the notions of well-posedness and evolution systems have to be slightly modified: we add the requirement that $\{x(t, s, \,.\,) : (s, t) \in \Delta_J\}$ be equicontinuous to the definition of well-posedness and the requirement that $\{U(t, s) : (s, t) \in \Delta_J\}$ be equicontinuous to the definition of evolution systems. (When X is a Banach space these extra conditions follow from the original ones, so that the new definition is consistent with the old one.) With these adjustments Proposition 2 carries over to the case of sequentially complete locally convex spaces X (over $\mathbb{C}$) as can be easily seen.

As a final preliminary we introduce the notion of (M, ω)-stability which was devised by Kato in his article [10].

Suppose that $A(t) : D(A(t)) \subset X \to X$ is a densely defined linear operator for every $t \in J = [a, b]$ and let $M \in [1, \infty)$ and $\omega \in \mathbb{R}$. The family A is then called (M, ω)-*stable* if and only if $(\omega, \infty) \subset \rho(A(t))$ for all $t \in J$ and

$$\left\|(\lambda - A(t_n))^{-1} \cdots (\lambda - A(t_1))^{-1}\right\| \leq \frac{M}{(\lambda - \omega)^n}$$

for all $\lambda \in (\omega, \infty)$ and all $t_1, \dots, t_n \in J$ satisfying $t_1 \leq \dots \leq t_n$ (with arbitrary $n \in \mathbb{N}$).

It is well-known (from Kato's article [10]) that A is (M, ω)-stable if and only if each $A(t)$ generates a strongly continuous semigroup on X and

$$\left\|e^{A(t_n)s_n} \, \cdots \, e^{A(t_1)s_1}\right\| \leq M e^{\omega(s_1 + \cdots + s_n)}$$

for all $s_1, \dots, s_n \in [0, \infty)$ and all $t_1, \dots, t_n \in J$ satisfying $t_1 \leq \dots \leq t_n$ (with arbitrary $n \in \mathbb{N}$)). In the special case of normal generators, (M, ω)-stability is equivalent to

$(1,\omega)$-stability which in turn is equivalent to the condition that every member $A(t)$ of the family generates a quasicontractive semigroup whose quasicontractive growth bound is less than or equal to ω (where we define the quasicontractive growth bound of a semigroup to be the infimum of all real numbers ω' such that the semigroup is dominated by the function $s \mapsto e^{\omega' s}$). In general, however, (M,ω)-stability and $(1,\omega)$-stability do not coincide which can be seen, for example, by setting $A(t) = A_0$ for every $t \in J$ and choosing A_0 as in Example I.5.7 (iii) of [7].

In case X is a locally convex space with a generating familiy P of seminorms, we say that A is $\big((M_p)_{p\in P},\omega\big)$*-stable* (where $M_p \in [1,\infty)$ for every $p \in P$ and $\omega \in \mathbb{R}$) if and only if $\lambda - A(t)$ is boundedly invertible for every $\lambda \in (\omega,\infty)$ and every $t \in J$, and

$$p\Big((\lambda - A(t_n))^{-1} \cdots (\lambda - A(t_1))^{-1}\, x\Big) \le \frac{M_p}{(\lambda-\omega)^n}\, p(x)$$

for all $\lambda \in (\omega,\infty)$ and all $t_1,\dots,t_n \in J$ satisfying $t_1 \le \dots \le t_n$ (with arbitrary $n \in \mathbb{N}$)) and all $x \in X$.

3 Why can Yosida's theorem be simplified ...

3.1 ... in the case of normed spaces?

We now cite Yosida's sufficient conditions for well-posedness that we are going to simplify in the sequel. In the case $M = 1$ these are just the hypotheses of Theorem XIV.4.1 of [27]. In the following, we shall always write I for the closed unit interval $[0,1]$ and Δ for the triangular set $\Delta_I = \{(s,t) \in I^2 : s \le t\}$.

Assumption 3. *$A(t) : D \subset X \to X$ is, for every $t \in I$, a boundedly invertible densely defined linear map such that A is $(M,0)$-stable and such that the following holds true:*

(i) $\{(s',t') \in I^2 : s' \neq t'\} \ni (s,t) \mapsto \frac{1}{t-s}\, C(t,s)x$ *is bounded and uniformly continuous for all $x \in X$, where $C(t,s) := A(t)A(s)^{-1} - 1$*

(ii) $C(t)x := \lim_{k\to\infty} k\, C(t, t-\frac{1}{k})x$ *exists uniformly in $t \in (0,1]$ for all $x \in X$*

(iii) $(0,1] \ni t \mapsto C(t)x$ *is continuous for all $x \in X$.*

It follows from Yosida's proof in [27] (which also works – mutatis mutandis – in the case where $M \in (1,\infty)$) that under the above assumptions there is a unique evolution system U for A on D and the following holds true:

$$\|U(t,s)\| \le M \quad \text{for all } (s,t) \in \Delta.$$

Yosida's proof really gives an evolution system for A on D (although not all properties encapsulated in the notion of an evolution system for A are stated in his theorem). In particular, his proof shows that $t \mapsto U(t,s)x$ is continuously differentiable for all $x \in D$.

We remark that the conclusion of Yosida's theorem just mentioned still holds true if the original assumptions are replaced by the following modified – and a priori weaker – assumptions (where the uniform continuity requirement in (i) is omitted and the requirement that $(0,1] \ni t \mapsto C(t)x$ be continuously extendable to the left endpoint 0 is added):

Assumption 4. *$A(t) : D \subset X \to X$ is, for every $t \in I$, a boundedly invertible densely defined linear map such that A is $(M,0)$-stable and the following holds true:*

(i) $\{(s',t') \in I^2 : s' \neq t'\} \ni (s,t) \mapsto \frac{1}{t-s} C(t,s)x$ is bounded for all $x \in X$, where $C(t,s) := A(t)A(s)^{-1} - 1$

(ii) $C(t)x := \lim_{k\to\infty} k\, C(t, t-\frac{1}{k})x$ exists uniformly in $t \in (0,1]$ for all $x \in X$

(iii) $(0,1] \ni t \mapsto C(t)x$ is continuous and continuously extendable to the left endpoint 0 for all $x \in X$.

It is the thus modified assumptions that are really used in Yosida's proof (as can be explicitly seen, for instance, from the exposition in [21] (Lemma 3.12 and the remark following its proof)). And it is therefore natural to take into consideration these modified assumptions as well.

We finally consider the following very simple conditions.

Assumption 5. *$A(t) : D \subset X \to X$ is, for every $t \in I$, a boundedly invertible densely defined linear map such that A is $(M,0)$-stable and $t \mapsto A(t)x$ is continuously differentiable for all $x \in D$.*

It follows from the main theorem (Theorem 4) of Kato's seminal article [9] that under the above assumptions there is a unique evolution system U for A on D and U satisfies the estimate $\|U(t,s)\| \leq M$ for all $(s,t) \in \Delta$. And by a simple translation argument one sees that the same conclusion holds true if one omits from Assumption 5 the requirement that each $A(t)$ be boundedly invertible.

We can now state the main result of this note revealing that the hypotheses of Yosida's theorem (Assumption 3) are equivalent to – and can therefore be replaced by – the much simpler conditions above (Assumption 5). In particular, this theorem shows that Yosida's theorem (Theorem XIV.4.1) of [27] is equivalent to Kato's theorem (Theorem 4) from [9].

Theorem 6. *Assumption 3, Assumption 4 and Assumption 5 are equivalent to each other.*

Proof. Suppose that Assumption 3 is satisfied. It follows, by virtue of the uniform continuity condition in (i) and the uniform convergence condition in (ii) of Assumption 3, that for every $x \in X$

$$\sup_{t\in[h,1]} \left\| \frac{1}{h} C(t,t-h)x - C(t)x \right\| \longrightarrow 0 \quad (h \searrow 0).$$

And from this in turn one infers, invoking the uniform continuity condition in (i) again, that $(0,1] \ni t \mapsto C(t)x$ is (continuous and) continuously extendable to the left endpoint 0 for every $x \in X$. (In particular, this shows that condition (iii) of Assumption 3 is implicit in conditions (i) and (ii) and could therefore be omitted from Yosida's original assumptions.) Thus, Assumption 4 is fulfilled.

Suppose now that Assumption 4 is satisfied and let $x \in D$. We show that the map $t \mapsto f(t) = A(t)x$ satisfies the hypotheses of Lemma 1. It follows from the boundedness condition in (i) of Assumption 4 that f is continuous: indeed, for every $t \in [0,1]$, one has

$$f(t+h) - f(t) = \Big(A(t+h)A(t)^{-1} - 1\Big)A(t)x = C(t+h,t)A(t)x \longrightarrow 0 \quad (h \to 0).$$

Second, one deduces from the uniform convergence condition in (ii) of Assumption 4 and the continuity of $t \mapsto A(t)x$ just proved that for every $t \in (0,1]$

$$-k\Big(f\big(t-\frac{1}{k}\big) - f(t)\Big) = k\, C\big(t, t-\frac{1}{k}\big)A\big(t-\frac{1}{k}\big)x \longrightarrow C(t)A(t)x \quad (k \to \infty),$$

and that this convergence is even uniform in $t \in (0,1]$. And third, the limit map $(0,1] \ni t \mapsto C(t)A(t)x$ is continuous and continuously extendable to the left endpoint 0 by condition (iii) of Assumption 4 and the continuity of f. We have thus verified all hypotheses of Lemma 1, and this lemma shows that Assumption 5 is fulfilled.

Suppose finally that Assumption 5 is satisfied. Then $s \mapsto A(s)A(0)^{-1}$ is strongly continuously differentiable and hence norm continuous and therefore also its inverse $s \mapsto \big(A(s)A(0)^{-1}\big)^{-1} = A(0)A(s)^{-1}$ is norm continuous, in particular, strongly continuous. Thus, the map

$$I^2 \ni (s,\tau) \mapsto A'(\tau)A(s)^{-1}x = A'(\tau)A(0)^{-1}\, A(0)A(s)^{-1}x$$

is continuous for every $x \in X$. And from this, using the integral representation

$$\frac{1}{t-s}\, C(t,s)x = \frac{1}{t-s}\, \big(A(t) - A(s)\big)A(s)^{-1}x = \frac{1}{t-s} \int_s^t A'(\tau)A(s)^{-1}x \, d\tau$$

(valid for all $x \in X$ and all $(s,t) \in \{s' \neq t'\}$), one readily obtains that $\{s' \neq t'\} \ni (s,t) \mapsto \frac{1}{t-s}\, C(t,s)x$ extends to a continuous map on the whole of I^2 from which conditions (i) through (iii) of Assumption 3 are obvious. Thus, Assumption 3 is fulfilled. ■

3.2 ... in the case of locally convex spaces?

We now comment on Yosida's theorem for locally convex spaces from the article [26]. It gives sufficient conditions for the well-posedness on D of the initial value problems corresponding to a family A of linear operators $A(t) : D \subset X \to X$ with common dense domain D in a locally convex space X. In detail, the hypotheses are as follows:

Assumption 7. *$A(t) : D \subset X \to X$ is, for every $t \in I$, a boundedly invertible densely defined linear map in a sequentially complete locally convex space X over $\mathbb{C}$. A is $\big((M_p)_{p \in P}, 0\big)$-stable for a family P of seminorms on X generating the topology of X, and the following holds true:*

(i) $\{(s',t') \in I^2 : s' \neq t'\} \ni (s,t) \mapsto \frac{1}{t-s}\, C(t,s)x$ *is bounded and uniformly continuous for all* $x \in X$*, and for every* $p \in P$ *there is a constant* $c_p \in [0,\infty)$ *such that* $p(\frac{1}{t-s}\, C(t,s)x) \leq c_p\, p(x)$ *for all* $(s,t) \in \{s' \neq t'\}$ *and all* $x \in X$ *(where* $C(t,s) := A(t)A(s)^{-1} - 1$ *as above)*

(ii) $C(t)x := \lim_{k\to\infty} k\, C(t, t - \frac{1}{k})x$ *exists uniformly in* $t \in (0,1]$ *for all* $x \in X$

(iii) $C(t)$ *is a bounded linear map for every* $t \in (0,1]$*.*

With the exception of the seminorm estimate in (i), all conditions above are well-expected from Yosida's theorem for Banach spaces. (At first, one might miss in (iii) the additional requirement that $(0,1] \ni t \mapsto C(t)x$ be continuous for all $x \in X$, but this follows from the other conditions as it did in the case of Banach spaces (see the proof of Theorem 6).) Similarly to the Banach space case, the hypotheses in the case of locally convex spaces can be simplified (although not as remarkably as in the former case).

Assumption 8. $A(t) : D \subset X \to X$ *is, for every* $t \in I$*, a boundedly invertible densely defined linear map in a sequentially complete locally convex space* X *over* $\mathbb{C}$*.* A *is* $\big((M_p)_{p\in P}, 0\big)$*-stable for a family* P *of seminorms on* X *generating the topology of* X*,* $t \mapsto A(t)x$ *is continuously differentiable for all* $x \in D$*, and for every* $p \in P$ *there is a constant* $c_p \in [0,\infty)$ *such that* $p(\frac{1}{t-s}\, C(t,s)x) \leq c_p\, p(x)$ *for all* $(s,t) \in \{s' \neq t'\}$ *and all* $x \in X$ *(which is satisfied if, for instance,* $p(A'(t)A(s)^{-1}x) \leq c_p\, p(x)$ *for all* $(s,t) \in I^2$ *and all* $x \in X$*).*

We have the following equivalence generalizing the equivalence statement of Theorem 6.

Theorem 9. *Assumption 7 and Assumption 8 are equivalent to each other.*

We allow ourselves to omit the proof of this theorem since it roughly follows the lines of the proof of Theorem 6 above. A word concerning the seminorm estimate accompanying the (well-expected) continuous differentiability condition seems in order, however: at first, one might think that this seminorm estimate should be replaceable by the handier requirement that $\{s' \neq t'\} \ni (s,t) \mapsto \frac{1}{t-s}\, C(t,s)x$ be bounded – at least in case X is additionally assumed to be barreled (in which case the uniform boundedness theorem of Banach and Steinhaus is available). Yet this is, in general, not possible since, in order for Yosida's proof to work, it is essential to have on the right hand side of the estimate the same seminorm as on the left – and not just any continuous seminorm depending on the seminorm on the left (which the uniform boundedness theorem would give, indeed).

4 Concluding remarks

Although the equivalences proved in the previous section were not really hard to see – basically, we had only to realize that Lemma 1 can be applied – they seem to be new. At least, they cannot be found in the standard textbooks by Yosida [27], Reed and Simon [18], Krein [15], Tanabe [23], or Pazy [17] – nor can they be found in Kato's articles [10], [11] (or generalizations of them such as [14] or [8]) or more recent review articles

on well-posedness of non-autonomous linear evolution equations such as Schnaubelt's article [22].

We conclude by commenting on how Yosida's theorem is related to other classical results on well-posedness. A first consequence of the equivalences proved in the previous section is that Yosida's theorem (for Banach spaces) is contained as a special case in the first main result (Theorem 3.0) of Kisyński's article [13] which – just as Yosida's theorem – deals with families of operators $A(t)$ with time-independent domain D and gives sufficient conditions for the initial value problems corresponding to A to be well-posed on D. In detail, these sufficient conditions are equivalent to the following (where equivalence is due to the fact that $(M,0)$-stability can be characterized – by Proposition 1.3 of [16] – in terms of monotonic families of norms as employed by Kisyński): $A(t) : D \subset X \to X$ is, for every $t \in I$, a linear map such that A is $(M,0)$-stable and $t \mapsto A(t)x$ is weakly continuously differentiable for every $x \in D$ (that is, for every $x^* \in X^*$ the function $t \mapsto x^*(A(t)x)$ is continuously differentiable and, in addition, there are vectors $y_x(t)$ such that $\frac{d}{dt}\big(x^*(A(t)x)\big) = x^*(y_x(t))$ for all $t \in I$). It should be pointed out that Kisyński's theorem is strictly more general than Yosida's theorem: there really are linear operators $A(t) : D \subset X \to X$ meeting the above requirements of Kisyński's theorem but failing to satisfy those of Yosida's theorem. Indeed, take $A(t) := A + B(t) - b$, where $A : D \subset X \to X$ generates a contraction semigroup on X, the $B(t)$ are bounded linear operators in X such that $t \mapsto B(t)$ is continuously differentiable with respect to the weak operator topology but not with respect to the strong operator topology, and $b := \sup_{t \in I} \|B(t)\|$. X could be chosen to be $L^2(\mathbb{R})$ and $B(t)$ could be defined by $B(t)g := \int_0^t \langle g_0, g \rangle\, f(\tau)\, d\tau$, where $f(t)(x) := f_0(x)\, e^{\frac{i\,x}{t}}$ for $t \in (0,1]$ and $x \in \mathbb{R}$, $f(0) := 0$, and f_0, g_0 are arbitrary non-zero elements of X. It should also be pointed out, however, that there are some general connections between weak and strong regularity revealing that, roughly speaking, the former is not much more general than the latter. See, for example, the paper [19] and the references therein for precise statements.

A second consequence of the equivalences established in the previous section is that Yosida's theorem (for Banach spaces) is contained as a special case in the main result (Theorem 6.1) of Kato's article [10]. In this theorem linear operators $A(t) : D(A(t)) \subset X \to X$ with time-dependent domains are considered and sufficient conditions for the well-posedness of the initial value problems corresponding to A on a certain subspace Y of X are given.

And, finally, since the main theorem (Theorem 1) of Kato's article [11] generalizes both Theorem 6.1 of [10] and Theorem 3.0 of [13] (as can be readily seen), it a fortiori contains Yosida's theorem as a special case.

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